

GEOMETRIC CHARACTERIZATION OF SOLITONS

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ABSTRACT. I show that H^1 solutions of the nonlinear Schroedinger equation which are incoming converge to a soliton, in the radial case.

SECTION 1

1. Introduction.

The aim of this work is to describe a geometric definition of localized solutions of NLS.

In the linear case we have the RAGE Theorem, which relates localized solutions to the pure point spectrum of the Hamiltonian: Localized solutions of the linear Schroedinger equation are linear combinations of L^2 eigenfunctions of the Hamiltonian. In particular, they are almost periodic functions of time. For the nonlinear case see [Sig, Sof, Tao].

The question arises as to what is the analog of the bound states of a linear equation, in the nonlinear case.

Here, I will show that solitons appear naturally from geometric considerations. It lends support to the conjecture that all generic outgoing states of NLS are solitons and free waves. That is, I will show that if the solution of NLS is purely incoming, up to $L^1(dt)$ corrections, then the solution converges to a soliton, in any compact region around the origin.

The method of proof is based on and motivated by the hydrodynamic reformulation of the Schroedinger equation.

The incoming wave condition is then written in terms of the notion of flux through surfaces around the origin.

It is then shown how to rigorously use the hydrodynamic formulation, by restricting the analysis to topologically trivial domains of space-time where the solution is nonvanishing.

The solution in such regions can then be uniquely written in the polar form, with continuous phase function. This, together with the a-priori H^1 bound is then

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used to construct the velocity function, and make sense of the related Euler type equation it satisfies. It should be noted that here we do not use the semiclassical limit, so the "Quantum Potential" term is not ignored.

Consider the NLS in 1 or 3 dimensions (for simplicity):

$$(1.1) \quad i \frac{\partial \psi}{\partial t} = -\Delta \psi + F(|\psi|)\psi$$

$\psi(t=0) = \psi_0 \in H_{\text{radial}}^1 \cap L^2(\mathbb{R}^n, |x|d^n x)$ that is

$$(1.2) \quad \| |x| \psi_0 \|_{L^2} + \| \nabla \psi_0 \|_{L^2} + \| \psi_0 \|_{L^2} < \infty.$$

The nonlinearity $F(|\psi|)$ is assumed to be of the RSS type [RSS] which guaranties global existence and stable soliton (ground state) solutions, again for simplicity. The property of stability is not used.

We will now make the following main time-dependent a-priori assumptions on the solutions of (1) with ψ_0 as initial data:

H^1 (Energy) Boundedness.

$$(1.3) \quad \| \psi(t) \|_{H^1} \leq E < \infty \text{ uniformly in } t.$$

This assumption is of course verified whenever global existence is proved.

Incoming Wave Condition (IWC).

$$(1.4) \quad i \hat{r} \cdot (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) \leq 0 \text{ for all times, any } r.$$

Here $\hat{r} = \vec{r}/r = x/|x|$. The above inequality is to be understood as holding almost everywhere in space. That this notion makes sense follows since the above expression is an L^1 function by (1.3).

This condition will be relaxed in various ways later. We now combine the above two conditions with the energy and dilation identities:

$$(1.5a) \quad \frac{d}{dt} \left\{ \frac{1}{2} \int_{\mathbb{R}^n} |\nabla \psi|^2 + \int \tilde{F}(|\psi|) |\psi|^2 \right\} = 0$$

$$(1.5b) \quad \frac{d}{dt} \| \psi \|_{L^2(\mathbb{R}^n)}^2 = 0$$

(1.5c)

$$\frac{d}{dt} \| |x| \psi \|_{L^2(\mathbb{R}^n)}^2 = (\psi, i[-\Delta, |x|^2] \psi) = 2(\psi, (-i\nabla_x \cdot x - ix \cdot \nabla_x) \psi) = 4(\psi, A\psi) \leq 0.$$

(1.6)

$$\text{where } A \equiv (-ix \cdot \nabla_x - i\nabla_x \cdot x)/2$$

where we used the IWC at the last step.

$\tilde{F}(|\psi|)$ is obtained from $F(|\psi|)$ in a known way. From the last identity we get

(1.7)

$$\sup_t \int |x\psi|^2 d^n x \leq \int |x\psi_0|^2 d^n x < \infty.$$

Since we also have $\psi \in H^1$, $\sup_t \|\psi\|_{H^1} < E$ it follows

Proposition 1.1.

- a) The trajectory $\{\psi(t)\}_{t=0}^\infty$ is precompact in H^s , for all $0 \leq s < 1$.
- b) Define $|\psi|^2 = \rho$ and $|\psi| = \sqrt{\rho} = \eta$ then, since

$$|\nabla|\psi|| \leq |\nabla\psi| \text{ a.e.,}$$

$$\{|\psi(t)| = \eta(t)\}_{t=0}^\infty \text{ is precompact in } H^s, 0 \leq s < 1.$$

Remarks

- We will repeatedly use the fact that weak convergence of η_t implies strong convergence in H^s , $0 \leq s < 1$, and in L^p , $2 \leq p < \frac{2n}{n-2}$, due to the uniform H^1 bound.

- The above arguments extend to any monotonic increasing function of r replacing r in the dilation identity, with appropriate assumption on the localization of ψ_0 :

$$\| |x|^\delta \psi_0 \| < \infty. \quad 0 < \delta.$$

SECTION 2 - CONVERGENCE OF ρ_t, η_t

The vector field \underline{v} .

For ψ as above consider the current density \vec{J} .

(2.1)

$$\vec{J} = -\frac{i}{2}(\bar{\psi}\nabla\psi - \psi\nabla\bar{\psi}).$$

Since $\psi, \nabla\psi$ are in L^2 by our H^1 bound, \vec{J} is in L^1 , uniformly in t . For $(x, t) \in \mathbb{R}^n \times \mathbb{R}$ such that $\psi(x, t) \neq 0$, we can define

(2.2)

$$|\psi|^{-2} \vec{J} = \rho^{-1} \vec{J} = \underline{v}(x, t).$$

$\tilde{v}(x, t)$ is defined on the set $Q = \{(x, t) | \psi(x, t) \neq 0\}$. We also denote, for each t , by N_t the set where $\psi(x, t) = 0$. In one dimension ψ is bounded and continuous. Moreover, Q is open, N_t^c is open. So, for each point $(x, t) \in Q$, we can find a neighborhood U such that $|\psi(x', t')| > \delta, (x', t') \in U$, by (joint) continuity, see below.

Since $\vec{J} \in L^2(\mathbb{R})$, it follows that $|\psi|^{-2}\vec{J}$ is $L^2_{\text{loc}}(U)$ so \tilde{v} is well defined on Q , as a function.

Next, we extend the definition of \tilde{v} to all of $\mathbb{R}^n \times \mathbb{R}$ by 0:

$$(2.3) \quad \underline{v} \equiv \begin{cases} \tilde{v} & \text{on } Q \\ 0 & \text{on } Q^c \end{cases}.$$

Next, since $\nabla_{(x,t)}\psi = (\nabla\psi, \partial_t\psi) = 0$ a.e. on Q^c it follows that

$$(2.4) \quad \vec{J} = \rho \underline{v} \text{ a.e. on } \mathbb{R}^n \times \mathbb{R},$$

and moreover we can compute on Q , where $\frac{\psi}{|\psi|}$ is bounded and continuous as a map from Q to S^1 :

$$\begin{aligned} \nabla \frac{\psi}{|\psi|} &= \eta^{-1} \nabla \psi - \eta^{-2} \psi \nabla |\psi| \\ &= \eta^{-1} \nabla \psi - \eta^{-2} \psi \frac{1}{2} \eta^{-1} (\bar{\psi} \nabla \psi + \psi \nabla \bar{\psi}) \\ &= \eta^{-1} \bar{\psi}^{-1} \frac{1}{2} [\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi}] \\ \nabla \psi &= \nabla \left(\eta \frac{\psi}{|\psi|} \right) = \bar{\psi}^{-1} \frac{1}{2} [\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi}] \\ &\quad + \frac{\psi}{|\psi|} \nabla \eta \\ \frac{\bar{\psi}}{|\psi|} \nabla \psi &= \eta^{-1} \frac{1}{2} [\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi}] + \nabla \eta \\ &= i \eta \underline{v} + \nabla \eta. \end{aligned}$$

For each t ,

$$\begin{aligned} (2.5) \quad \int_{\mathbb{R}} |\nabla \psi|^2 &= \int_{N_t^c} |\nabla \psi|^2 = \int_{N_t^c} \left| \frac{\bar{\psi}}{\psi} \nabla \psi \right|^2 = \int_{N_t^c} |\nabla \eta|^2 + \int_{N_t^c} \eta^2 |\underline{v}|^2 \\ &= \int_{\mathbb{R}} |\nabla \eta|^2 + \int_{\mathbb{R}} \eta^2 |\underline{v}|^2 \end{aligned}$$

since $\nabla\eta$ and $\eta\underline{v}$ are zero a.e. on N_t . We therefore have that $\underline{v} \in L^2(\eta^2 dx)$.

Next, we construct the angular velocity function ω , in a similar way.

Again, on Q we have

$$\begin{aligned}
\partial_t \frac{\psi}{|\psi|} &= \frac{1}{|\psi|} \dot{\psi} - \frac{\psi}{|\psi|^2} \partial_t |\psi| \\
&= |\psi|^{-1} [\dot{\psi} - \frac{\psi}{|\psi|} \frac{1}{2} |\psi|^{-1} (\bar{\psi} \dot{\psi} + \dot{\bar{\psi}} \psi)] \\
&= |\psi|^{-1} [\frac{1}{2} \dot{\psi} - \frac{1}{2} \frac{\psi}{|\psi|^2} \dot{\bar{\psi}} \psi] \\
&= \eta^{-1} \frac{1}{2} [\dot{\psi} - \frac{\psi}{\dot{\bar{\psi}}} \dot{\bar{\psi}}] \\
&= \frac{1}{2} \eta^{-1} [\bar{\psi} \dot{\psi} - \psi \dot{\bar{\psi}}] \frac{1}{\dot{\bar{\psi}}}.
\end{aligned}$$

Let, on Q :

$$\begin{aligned}
(2.6) \quad \omega &\equiv \frac{\bar{\psi}}{|\psi|} \partial_t \frac{\psi}{|\psi|} = \eta^{-2} \frac{-i}{2} [\bar{\psi}(i\dot{\psi}) + \psi(i\dot{\bar{\psi}})] \\
&= \eta^{-2} (-\frac{i}{2}) [\bar{\psi}(-\Delta\psi) + \psi(-\Delta\bar{\psi})] + \eta^{-2} O(F(|\psi|)).
\end{aligned}$$

Then, since $\eta^{-2}\bar{\psi}$ is continuous in Q , and

$$-\Delta\psi \in H^{-1}(\mathbb{R})$$

ω is a distribution, for each t .

Also, since η^2 is bounded, continuous, $\nabla\eta \in L^2$,

$$\eta^2\omega \in H^{-1}(\mathbb{R}).$$

Moreover, ω is trivially a distribution in $H^{-1}(Q)$.

Proposition. (*Joint Continuity*) Under the conditions of section I on ψ , $\psi(x, t)$ is jointly continuous in (x, t) in one dimension.

The same holds in $\mathbb{R}^3/\{0\} \times \mathbb{R}$.

Proof.

$$(A1) \quad \psi(x, t) = e^{i\Delta t} \psi_0 - i \int_0^t e^{i\Delta(t-s)} F(|\psi(s)|) \psi(s) ds$$

$$(A2) \quad \nabla \psi(x, t) = e^{i\Delta t} \nabla \psi_0 - i \int_0^t e^{i\Delta(t-s)} [G(\psi)\psi + F(|\psi|)\nabla \psi] ds$$

$$(A3) \quad |\psi(x, t) - \psi(x, t')| \leq c \|\psi(t) - \psi(t')\|_{H^{1/2+\varepsilon}}.$$

Therefore, to prove continuity, it is sufficient to prove continuity in t into $H^1(\mathbb{R})$. For this we use (A2). Since $\nabla \psi_0 \in L^2$, the Spectral and Von-Neumann theorem gives continuity in t of the first term (in L^2).

As for the second term, continuity follows if the integrand in A2 is $L^1(L^2(\mathbb{R}), dt)$, that is, if

$$(2.7) \quad \|G(\psi)\psi + F(|\psi|)\nabla \psi\|_{L^2(\mathbb{R})} \in L^1(dt).$$

Since ψ is bounded and $\nabla \psi \in L^2$, the result follows. Similar computations work in the radial case in three dimensions. \square

In the η, \underline{v} representation the IWC becomes:

When $\rho \neq 0$:

$$(2.8) \quad i\hat{r} \cdot (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) = 2\rho \hat{r} \cdot \underline{v} = 2\eta^2 \hat{r} \cdot \underline{v}.$$

When $\rho = 0$, $\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi = 0$ and hence we still have

$$(2.9) \quad i\hat{r} \cdot (\psi \nabla \bar{\psi} - \bar{\psi} \nabla \psi) = 2\eta^2 \hat{r} \cdot \underline{v} (= 0).$$

so that

$$(2.10) \quad \text{for all } x, t \quad \text{IWC: } \eta^2 \hat{r} \cdot \underline{v} \leq 0.$$

Furthermore, the H^1 bound gives

$$(2.11a) \quad \sup_t \|\nabla \eta\|_{L^2}^2 \leq \mathcal{E}_\rho < \infty$$

$$(2.11b) \quad \sup_t \|\eta \underline{v}\|_{L^2}^2 \leq \mathcal{E}_v < \infty$$

using equation (8).

We can now use Sobolev estimates to conclude that

Proposition 2.1. *For $n = 1, \eta$ is bounded and continuous function of x . For $n = 3, \eta(r)$ is bounded and continuous, for $r \neq 0$. The bounds are uniform in time, t .*

The Schrödinger equation gives:

$$(2.12) \quad \eta \dot{\eta} = \frac{1}{2} \partial_t (\eta^2) = \frac{1}{2} \partial_t \rho = 2\eta \eta' \cdot \underline{v} + \eta^2 \nabla \cdot \underline{v}.$$

Using this, we have

$$\frac{1}{2} \partial_t \int_{R_1 \leq r \leq R_2} \eta^2 d^n x = \int_{R_1 \leq r \leq R_2} [\nabla(\eta^2) \cdot \underline{v} + \eta^2 \nabla \cdot \underline{v}] d^n x = \int_S \eta^2 \underline{v} \cdot d\vec{S}$$

where S is the closed surface of the domain $R_1 \leq r \leq R_2$. For $n = 1$, S is $\{R_1, R_2\}$. Let \hat{N} be the unit normal vector to surface S . Using radial symmetry we arrive at

Lemma 2.1.

a) *For almost all t ,*

$$\begin{aligned} \frac{1}{2} \partial_t \int_{R_1 \leq r \leq R_2} \rho d^n x &= c_n(r^{n-1} \eta^2(r, t) \underline{v} \cdot \hat{N}) \Big|_{r=R_1}^{R_2} \\ &= c_n(R_2^{n-1} \eta^2(R_2, t) \underline{v} \cdot \hat{N}(R_2, t) - R_1^{n-1} \eta^2(R_1, t) \underline{v} \cdot \hat{N}(R_1, t)). \end{aligned}$$

b) *The above formula holds for all t , almost everywhere in $R_i, i = 1, 2$.*

Lemma 2.2 (Flux at infinity).

Under our assumptions, the flux $\eta^2 \underline{v} \cdot \hat{N}$ vanishes at infinity.

Proof. Since $\psi \in H^1$, we know that

$$\int \eta^2 |\underline{v}|^2 d^n x < \infty \text{ which implies the existence of a sequence } r_m \rightarrow \infty, \text{ such that}$$

$$r_m^{n-1} \eta^2(r_m) |\underline{v}(r_m)|^2 \rightarrow 0 \text{ as } r_m \rightarrow \infty,$$

for each t fixed.

Therefore, for each time t fixed,

$$(2.13) \quad r_m^{n-1} \eta^2(r_m) |\underline{v}(r_m)| \rightarrow 0,$$

otherwise $|\underline{v}(r_m)| \rightarrow 0$. But then, using the radial symmetry and the fact that $\eta \in H^1_{\text{radial}}$ it follows that

$$\begin{aligned} r^{n-1} \eta^2 &\leq c, \text{ which also implies that} \\ r_m^{n-1} \eta^2(r_m) |\underline{v}(r_m)| &\rightarrow 0. \end{aligned}$$

□

Lemma 2.3 (Flux at zero). *There is no flux through the origin*

Proof. By the previous Lemma,

$$(2.14) \quad \eta^2 \underline{v} \cdot \hat{N} \Big|_{r=R}^\infty = 0 - \eta^2 \underline{v} \cdot \hat{N} \Big|_{r=R} \leq 0.$$

Hence,

$$\partial_t \int_{0 \leq r \leq \infty} \eta^2 d^n x = 0 = \eta^2 \underline{v} \cdot \hat{N} \Big|_{R \rightarrow 0}^\infty = - \lim_{R \rightarrow 0} \eta^2 \underline{v} \cdot \hat{N} = 0.$$

□

From now on, on a sphere of radius R , we denote

$$\underline{v} \cdot \hat{N}(R, t) \equiv v_n(R, t).$$

Lemma 2.4. *For each $R, \eta^2 \hat{r} \cdot v(R, t) \in L^1(dt)$, uniformly in R .*

Proof.

$$(2.15) \quad \frac{1}{2} \int_0^T dt (\partial_t \int_{|x| \leq R} \eta^2 d^n x) = \frac{1}{2} \int_{|x| \leq R} \eta^2 d^n x \Big|_{t=0}^T = \int_0^T \eta^2 \hat{r} \cdot \underline{v}(R, t) dt$$

and since the last integrand is positive, by IWC, it follows that

$$\int_0^T |\eta^2 \hat{r} \cdot \underline{v}(R, t)| dt = \frac{1}{2} (\|E(|x| \leq R)\psi(T)\|_2^2 - \|E(|x| \leq R)\psi(0)\|_2^2) < \infty$$

since the left hand side is monotonic increasing in T , the result follows:

$$(2.16) \quad 0 \leq \int_0^T \eta^2 \hat{r} \cdot \underline{v}(R, t) dt \leq \frac{1}{2} \|\psi_0\|_2^2.$$

In particular, $\eta^2 v_n(R, t) \rightarrow'' 0''$ as an L^1 function. □

Proposition 2.6.

$$(2.17) \quad \text{Weak-} \lim_{t \rightarrow \infty} \eta(t) \equiv u, \text{ exists in } H^1.$$

Proof.

$$(2.18) \quad \int_{R_1 \leq |x| \leq R_2} \eta^2 d^n x \Big|_{t=0}^T = 2 \int_0^T dt \left(\eta^2 \hat{r} \cdot \underline{v} \Big|_{r=R_1}^{R_2} \right)$$

For each R_1, R_2 , the r.h.s. converges as $T \rightarrow \infty$, by Lemma (2.4). Hence, for each interval $I \subset \mathbb{R}$

$$\lim_{t \rightarrow \infty} \int \chi_I(r) \eta^2 d^n x$$

exists for χ_I the characteristic function of I .

The finite linear combinations of such $\chi_I, \sum \alpha_i \chi_{I_i}$ are dense in $L^p, 1 \leq p < \infty$.

Hence, given $\varepsilon > 0$, and $\varphi \in L^p, 1 \leq p < \infty$, we can find an element $\varphi_\varepsilon = \sum_{i=1}^N \alpha_i^\varepsilon \chi_{I_i}^\varepsilon, N < \infty$, and $\|\varphi - \varphi_\varepsilon\|_{L^p} \leq \varepsilon$.

So,

$$\begin{aligned} \lim_{t \rightarrow \infty} \int \varphi \eta^2 d^n x &= \lim_{t \rightarrow \infty} \int (\varphi - \varphi_\varepsilon) \eta^2 d^n x \\ &+ \lim_{t \rightarrow \infty} \int \varphi_\varepsilon \eta^2 d^n x. \end{aligned}$$

The first term on the r.h.s. is bounded by

$$\|\varphi - \varphi_\varepsilon\|_p \sup_t \|\eta^2\|_{L^{p'}} \leq C\varepsilon$$

for all $p' \leq \frac{n}{n-2} (p' \leq 3 \text{ for } n = 3, \infty \text{ in } 1 - \dim)$ and the second term converges as $t \rightarrow \infty$.

Hence, we have

Lemma. *If $\lim_{t \rightarrow \infty} \int \chi_I(r) \rho_t d^n x$ exists for all intervals I , then $\rho_t \xrightarrow{\omega} \rho_\infty$ in L^p , for any $p > 1$, provided*

$$\sup_t \|\rho_t\|_{L^p} < C < \infty.$$

In particular, in our case, since $\eta \in H^1$,

$$\begin{aligned} \eta^2 &\in L^q, 1 \leq q \leq 3 \text{ for } n = 3 \\ &1 \leq q \leq \infty \text{ for } n = 1, \end{aligned}$$

it follows that

$$(2.19) \quad \omega - \lim_{t \rightarrow \infty} \eta_t^2 \xrightarrow{L^q} u^2 \geq 0, u^2 \in L^q, q > 1.$$

Now let $\varphi \in C_0^\infty$ and the dimension to be 1. For all t, t' , we have

$$\int \varphi \nabla(\eta_t^2 - \eta_{t'}^2) = - \int (\nabla \phi)(\eta_t^2 - \eta_{t'}^2) \rightarrow 0 \text{ as } t, t' \rightarrow \infty$$

since η_t^2 is Cauchy in L^2 , say.

Since C_0^∞ is dense in L^2 and $\sup_t \|\nabla \eta_t^2\|_2 < \infty$, $\nabla \eta_t^2$ is Cauchy and hence converges in L^2 , weakly. Here we used that $\|\nabla \eta_t^2\|_2 \leq 2\|\eta_t\|_\infty \|\nabla \eta_t\|_2 \leq C\|\nabla \eta_t\|_2^2$. In three dimensions a similar argument applies:

$$\nabla(\eta_t^2) \in L^{3/2}, \text{ therefore}$$

$$\eta_t^2 \rightarrow u^2 \text{ weakly in } W^{1,3/2}.$$

Next, using that

$$\sup_t \int |x|^2 \rho_t d^n x < \infty$$

so that $\{\eta_t^2\}_{t \geq 0}$ is precompact in $H^s(\mathbb{R})$, $0 \leq s < 1$ and $(n = 3)$ in $W^{s,3/2}(\mathbb{R}^3)$ $0 \leq s < 1$.

Using that $W^{1,3/2} \hookrightarrow H^s$ for $s = 1/2$ ($n = 3$).

We conclude

Proposition.

$$\eta_t^2 \rightarrow u^2 \text{ strongly in } H^s(\mathbb{R}) \quad 0 \leq s < 1$$

and in $W^{s,3/2}$, $0 \leq s < 1$, $H^s(\mathbb{R}^3)$, $0 \leq s < 1/2$ in 3 dimensions.

Applying Sobolev embedding theorem again:

$$\eta_t^2 \rightarrow u^2 \text{ strongly in } L^p(\mathbb{R}^n), p < \frac{n}{n-2}$$

$n \geq 3$, and all p in 1 – dim.

Now, we use that to prove **weak** convergence of η_t in L^p .

Let $\varphi \in C_0^\infty$, supported in some compact interval $I \subset \mathbb{R}$,

$$|I| < \infty.$$

Then,

$$\begin{aligned} J \equiv \left| \int \varphi(\eta_t - u) \right| &\leq \int |\varphi \chi_1| (\eta_t - u)(\eta_t + u) |(\eta_t + u)^{-1}| \\ &\quad + \int |\varphi \chi_2| |\eta_t - u| \end{aligned}$$

where χ_1 is the characteristic function of the set where $\eta_t + u \geq \varepsilon > 0$, and $\chi_2 \equiv 1 - \chi_1$.

So,

$$J \leq \frac{1}{\varepsilon} \|\eta_t^2 - u^2\|_{L^p} \|\varphi\|_{L^{p'}} + \varepsilon |I|^{1/p} \|\varphi\|_{L^{p'}}$$

where we used that

$$\chi_2 |\eta_t - u| \leq \chi_2 (\eta_t + u) \leq \varepsilon \chi_2$$

since $\eta_t, u \geq 0$.

Given $\varepsilon_0 > 0$, choose ε s.t.

$$\varepsilon |I|^{1/p} \|\varphi\|_{L^{p'}} < \varepsilon_0/2.$$

For this ε , choose $T(\varepsilon \varepsilon_0)$ s.t. for all $t > T(\varepsilon \varepsilon_0)$

$$\|\eta_t^2 - u^2\|_{L^p} \|\varphi\|_{L^{p'}} \leq \varepsilon \varepsilon_0/2.$$

Hence $J < \varepsilon_0$. This can be done for all $2 \leq p < q_*$

$$\begin{aligned} q_* &= 3 \text{ when } n = 3 \\ q_* &= \infty \text{ when } n = 1. \end{aligned}$$

Hence the above applies to all $1 < p' \leq 2$ in 1 – dim and $3/2 < p' \leq 2$ in 3 – dim.

So, in $n = 1$, since η_t is uniformly bounded in L^p for all $p \geq 2$, $\eta_t \xrightarrow{L^p} u$ weakly for $2 \leq p < \infty$.

In three dimensions, η_t is uniformly bounded in L^p ,

$$2 \leq p \leq 6$$

so $\eta_t \xrightarrow{L^p} u$ weakly for $2 \leq p < 3$. We now take, as before $J \equiv |\int \varphi \nabla(\eta_t - \eta_{t'})|$ and argue as above, to conclude that

$$\nabla \eta_t \text{ is Cauchy in } L^2$$

which implies that $\nabla \eta_t \xrightarrow{L^2} \nabla \tilde{u}$ weakly. But then, for $\varphi \in C_0^\infty$, $\int \varphi \nabla(\eta_t - \tilde{u}) = -\int \nabla \varphi(\eta_t - \tilde{u}) \rightarrow 0$ implies that $\tilde{u} = u$, since $\eta_t \xrightarrow{L^2} u$ weakly. \square

As corollaries we have

Proposition 2.7.

$$\eta_t \xrightarrow{s} u \text{ in } H^s, 0 \leq s < 1.$$

PF

Follows from weak convergence in H^1 and from the precompactness in $H^s, s < 1$.

Proposition 2.9.

$$u \in H^1 \quad \|u\|_{L^\infty(\mathbb{R})} < \infty \quad u \in Lip(\frac{1}{2} - \varepsilon) \text{ in } \mathbb{R}.$$

$$|u| \leq c/\sqrt{r} \text{ in } H_{\text{rad}}^1(\mathbb{R}^3). \quad u \in Lip(\frac{1}{2}) \text{ away from zero in } \mathbb{R}^3.$$

SECTION 3 POINTWISE CONVERGENCE OF η, \underline{v} AND OTHER PROPERTIES

We have seen that $u \geq 0$, is continuous in \mathbb{R} , and in $\mathbb{R}^3/\{0\}$. Hence, if $u(x_0) > 0$, there is an interval \tilde{I} around x_0 where u vanishes for the first time at its end points, in one dimension. In three dimensions the same holds, provided the origin is outside $r \in \tilde{I}_{x_0}$. $r = |x|$. If $I \subset \tilde{I}_{x_0}$, I away from the boundary, then

$$u(x) > \delta > 0 \text{ for all } x \in I.$$

We will now analyze u in such I .

Proposition 3.1.

$$\eta_t \rightarrow u \text{ pointwise in } \mathbb{R} \text{ or } \mathbb{R}^3/\{0\} \text{ uniformly.}$$

Proof.

$$|\eta_t(x) - u(x)| \leq C \|\eta_t - u\|_{H^{1/2+\varepsilon}(\mathbb{R})} \rightarrow 0 \text{ as } t \rightarrow \infty$$

by proposition 2.7.

Similar estimate holds in $\mathbb{R}_{\text{rad}}^3/\{0\}$.

Lemma 3.2. *On I , the regularity of ψ carries over that of η .*

Proof. Say $s = 1 + \mu, \mu < 1$. Let $\psi = a + ib$.

$$\|D^s \rho\| = \|D^\mu(\bar{\psi} \nabla \psi + \psi \nabla \bar{\psi})\| \leq C \|D^\mu \psi\| \|\nabla \psi\| + C \|\psi\| \|D^{1+\mu} \psi\| < \infty$$

by assumption (that $\psi \in H^s$).

$$\|D^{1+\mu} \chi_I |\psi|\| = \|D^\mu D(\chi_I \sqrt{a^2 + b^2})\|$$

$$= \|D^\mu[(D\chi_I)|\psi|] + D^\mu\{\chi_I(a^2 + b^2)^{-1/2}(a\nabla a + b\nabla b)\}\|$$

so, we only need to bound

$$\|D^\mu[(a^2 + b^2)^{-1/2} \chi_I]\|_{L^q} \text{ for some } q \text{ large enough.}$$

This last expression is equal to

$$\|D^{-1+\mu} D[(a^2 + b^2)^{-1/2} \chi_I]\|_{L^q} \leq C D[(a^2 + b^2)^{-1/2} \chi_I]\|_{L^{\tilde{q}}}$$

for some $2 \leq \tilde{q} \leq 3$. But $D[(a^2 + b^2)^{-1/2} \chi_I] = -(a^2 + b^2)^{-3/2}(a\nabla a + b\nabla b)\chi_I + 0(1) \in L^{\tilde{q}}$ since $a, \chi_I(a^2 + b^2)^{-3/2} \in L^\infty, \nabla a, \nabla b \in L^{\tilde{q}}$. \square

Proposition 3.3.

Recall that $\rho \underline{v} \rightarrow'' 0''$ as $L^1(dt)$ for $t \rightarrow \infty$.

a) On $I \subset \tilde{I}$ we then have: ($\rho \equiv \eta^2$)

$$\eta \hat{r} \cdot \underline{v} \rightarrow'' 0'' \text{ and } \hat{r} \cdot \underline{v} \rightarrow'' 0'' \text{ as } L^1(dt).$$

b) $\chi_{\tilde{I}} \eta \hat{r} \cdot \underline{v} \xrightarrow{L^2} 0$ strongly and $\chi_I \hat{r} \cdot \underline{v} \xrightarrow{L^2} 0$ strongly, for $t_n \rightarrow \infty$.

Proof.

Part (a) follows, since on $I, \eta > \delta > 0$, uniformly in time, since on $I, u > \delta' \geq 2\delta$, and $\eta_t \rightarrow u$ pointwise by Proposition 3.1. To Prove (b), we note that $\|\eta \hat{r} \cdot \underline{v}\|_{L^2} < \infty$ uniformly in t , since

$$\eta^2 \hat{r} \cdot \underline{v} = \frac{i}{2} (\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi}) \cdot \hat{r} \text{ then } \eta \hat{r} \cdot v = \vec{J} \cdot \hat{r} / \eta (\text{ on } Q)$$

and

$$|\vec{J} / \eta| \leq |\nabla \psi| \in L^2,$$

and 0 otherwise, with uniform (in t) L^2 -norm.

We now use the following Lemma.

Lemma. *If $\|f_n\|_{L^p} < C < \infty$, $f_n \rightarrow f$ pointwise, then $\chi_J(f_n - f) \rightarrow 0$ in L^q , for all $q < p$ and J compact.*

By this Lemma the result (b) follows if we can prove that for a sequence $t_n \rightarrow \infty$,

$$\sup_n \|\chi_{\tilde{I}} \eta_{t_n} \hat{r} \cdot \underline{v}(t_n)\|_{L^q} < \infty$$

for some $q > 2$.

This follows from Strichartz estimate on compact time intervals, as it implies that $\|\nabla \psi_n\|_{L^3} \leq c \|\psi(0)\|_{H^1}$ uniformly in n , and

$$|\nabla \psi_n|^3 \geq |\eta_n \hat{r} \cdot \underline{v}_n|^3.$$

□

Proposition 3.4.

$$\chi_I \eta_n \rightarrow \chi_I u \text{ strongly in } H^s, s < 3/2.$$

Proof. The result follows from weak convergence and compactness of $\chi_I \eta_n$ in $H^s, s < 3/2$, if we can prove uniform boundedness in $H^s, s < 3/2$.

The weak convergence follows from $(\varphi \in C_0^\infty)$

$$\left| \int \varphi \chi_I D^s (\eta_n - \eta_m) \right| = \left| - \int (\varphi \chi_I)' D^{s-1} (\eta_n - \eta_m) \right|$$

$\leq C_\varphi \|D^{s-1}(\eta_n - \eta_m)\| \rightarrow 0$ as $n, m \rightarrow \infty$ by proposition 2.7 provided $0 \leq s-1 < 1$. So, the weak convergence will follow if $\|\chi_I \eta_n\|_{H^s} < c < \infty$ uniformly in n . For $s \leq 3/2$ this follows from Local smoothing estimates on compact time intervals. □

Corollary 3.1.

$$\chi_I u \in H^s, \quad s < 3/2$$

$$\chi_I u \in W^{1,3}.$$

SECTION 4 - THE EQUATION FOR u

Next, we would like to rewrite the NLS in an equivalent form, in terms of η, θ :

Let us compute, formally:

$$(4.1) \quad \begin{aligned} \psi &= \eta e^{-i\theta} & \nabla \psi &= (\nabla \eta) e^{-i\theta} - i\theta' \eta e^{-i\theta} \\ i\dot{\psi} &= i\dot{\eta} e^{-i\theta} + \dot{\theta} \eta e^{-i\theta} & F(|\psi|) &= F(\eta) \\ -\Delta \psi &= -\nabla \cdot (\eta' e^{-i\theta} - i\eta \theta' e^{-i\theta}) = -(\Delta \eta) e^{-i\theta} \\ &\quad + 2i\eta' \cdot \theta' e^{-i\theta} + \eta \theta'^2 e^{-i\theta} + i\eta \theta'' e^{-i\theta} \end{aligned}$$

where $'$ stands for gradient, $''$ for $\Delta : \eta' \cdot \theta' = \nabla \eta \cdot \nabla \theta, \theta'' = \Delta \theta$.

We have:

$$(4.2a) \quad \dot{\eta} = 2\eta' \theta' + \eta \theta'' = 2\nabla \eta \cdot \nabla \theta + \eta \Delta \theta$$

$$(4.2b) \quad -\Delta \eta + F(\eta) \eta = \dot{\theta} \eta - \eta \theta'^2.$$

Next, we embark on a way to make sense of these equations [see also AC, Ger, Gre, LZ] in some parts of the space time. Let $I \subset \tilde{I}$ be such an interval where $u \geq \delta > 0$ for all $r \in I$.

Since $\eta \rightarrow u$ uniformly on I (Proposition 3.4) we have that

$$(4.3) \quad \eta_t \geq \delta/2 > 0, \quad \forall t > T_0, \text{ on } I.$$

We therefore have that there exists a (good) box B in space-time, where $\eta \geq \delta/2 > 0$ for all $(x, t) \in B$,

$$B = \{x \in I, \quad t \geq T_0\}.$$

I is any interval strictly between two consecutive zeros of u . We always assume I does not contain the origin in \mathbb{R}^3 . We also recall the notation

$$\begin{aligned} Q &= \{(x, t) \in \mathbb{R}^n \times \mathbb{R}, \eta(x, t) \neq 0\} \\ N_t &= \{x \in \mathbb{R}^n, \eta(x, t) = 0\}. \end{aligned}$$

Since η is continuous in \mathbb{R} (or $\mathbb{R}^3/\{0\}$), the set N_t is closed, N_t^c is open.

Hence the function

$$(4.4) \quad \frac{\psi(x, t)}{|\psi(x, t)|} = \frac{\psi(x, t)}{\eta(x, t)} : Q \rightarrow S^1$$

is continuous (jointly in (x, t)).

For each t , N_t^c is a collection of open connected disjoint sets, $N_{t,i}^c, i = 1, 2, \dots$.

The key to constructing the phase function θ , with the right properties, is the existence of continuous extension to the (universal) covering space of S^1, \mathbb{R} .

The conditions for these are known:

Theorem. (*Lifting Lemma*)

Let $p : E \rightarrow B$ be a covering map of a topological space B , by E .

Let $p(e_0) = b_0$. Let $f : Y \rightarrow B$ be a continuous map of a topological space Y into B , with $f(y_0) = b_0$. Suppose that Y is path connected and locally path connected. The map f can then be lifted to a map $\tilde{f} : Y \rightarrow E$ such that

$$(4.5) \quad \tilde{f}(y_0) = e_0, \text{ if and only if}$$

$$(4.6 \text{ (Condition L)}) \quad f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(E, e_0))$$

Furthermore, if such a lifting exists, it is unique.

Here $\pi_1(A, a)$ denotes the first fundamental group of loops of A starting at the point a .

f_*, p_* are the induced maps on the loop spaces by f and p , respectively.

Using this theorem [GH] we can conclude the following :

- Since in general the space Q is not simply connected, and since $\pi_1(\mathbb{R}, e_0)$ is trivial, condition (L) will not be satisfied in general. So, the lifting of

$$f \equiv \frac{\psi}{\eta} : Q \rightarrow S^1$$

does not exist.

- For each t , and a connected part of N_t^c , a lifting $\mathcal{Q}_i(x, t)$ exists and is unique; but it is **not** continuous in t .

- A unique lifting $\theta_B(x, t)$ which is jointly continuous exists on good boxes B , since boxes have trivial fundamental group. It is this $\theta_B(x, t)$ which we will use from now on.

Using $\theta_B \equiv \theta$ we have on B :

$$(4.7) \quad \frac{\psi}{|\psi|} = e^{-i\theta(x, t)}, (x, t) \in B$$

with $\theta(x, t)$ jointly continuous in (x, t) .

Uniqueness is guaranteed, up to a choice of reference point, namely we choose a point (x_0, T_0) where $\frac{\psi}{|\psi|}$ is not on \mathbb{R}^- , and we choose $\frac{\psi}{|\psi|}$ to be on the first Riemann sheet of the $\ell n z$ function (with cut on \mathbb{R}^-).

We therefore established the following crucial identities

$$(4.8a) \quad \psi = \eta e^{-i\theta(x,t)}, (x, t) \in B$$

$$(4.8b) \quad \ell n \frac{\psi}{\|\psi\|} = -i\theta.$$

Remark. This construction is limited to the case of ψ being continuous.

Corollary. *Derivatives commute with lifting. Hence*

$$(4.9) \quad (f')_* = (f'_*)'.$$

To take a derivative of ψ requires the use of the chain rule (which uses the above Corollary).

$$(4.10) \quad \frac{\partial \psi}{\partial x} = (\partial_x \eta) e^{-i\theta} + -i\eta (\partial_x \theta) e^{-i\theta}$$

and similarly for $\frac{\partial \psi}{\partial t}$.

Once the above formula is established, it follows from the fact that $\nabla_x \psi, \partial_x \eta$ are in L^2 , that $\nabla_x \theta$ is also in L^2 since $\eta \geq \delta/2$ in B .

For $\dot{\theta}$, we use that $\dot{\theta}$ is well defined in the sense of distributions, hence

$$\int_B \int g(t) \varphi(x) \dot{\theta} dx dt = - \int_B \int g'(t) \varphi(x) \theta(x, t) dx dt.$$

Next, we compute higher derivatives, in the following weak sense: test functions $\varphi \in H_0^1(I)$.

Since $\frac{\partial \psi}{\partial x} \in L^2$, the above terms can be differentiated in the sense of distributions.

$$(4.11) \quad \begin{aligned} & \int \varphi \frac{\partial}{\partial x} [\eta' e^{-i\theta} - i\theta' \eta e^{-i\theta}] = \int \varphi \Delta \psi = \\ & = \int \varphi [\eta'' - i\theta'' \eta - i\theta' \eta' - i\theta' \eta' - \theta'^2 \eta] e^{-i\theta}. \end{aligned}$$

In fact, each term makes sense separately since $e^{-i\theta}$ is bounded, cont, η'', θ'' are derivatives of locally (square) integrable functions and so are distributions;

$$\theta'^2 \eta|_I = \eta^{-1} \theta'^2 \eta|_I^2 \leq (\delta/2)^{-1} (\eta \theta')^2 \in L^1_{\text{loc}}(I).$$

Similarly, $\theta' \eta'|_I \leq (\delta/2)^{-1} (\eta \theta') \eta' \in L^1_{\text{loc}}(I)$ since both η' and $\eta \theta'$ are in L^2_{loc} .

In particular

$$\begin{aligned} & \left| \int \varphi \theta'' \eta \right| = \left| - \int \varphi' \theta' \eta - \int \varphi \theta' \eta' \right| \\ & \leq \|\varphi\|_{H^1(I)} \left[\|\eta \theta'\|_{L^2(I)} + \left(\frac{\delta}{2} \right)^{-1} \|\eta'\|_{L^2} \|\eta \theta'\|_{L^2(I)} \right]. \end{aligned}$$

On B , we have

$$(4.12) \quad \int \varphi i \frac{\partial \psi}{\partial t} = \int i \varphi \frac{\partial \psi}{\partial t} = \int i \varphi \dot{\eta} e^{-i\theta} - 1 \int \varphi \eta \dot{\theta} e^{-i\theta}.$$

Let $\varphi \in C_0^\infty(\tilde{I})$. Then

Lemma 4.1.

$$(4.13) \quad (\varphi, -\Delta \eta) = (-\Delta \varphi, \eta) \rightarrow (-\Delta \varphi, u)$$

as $t \rightarrow \infty$.

This follows since $\eta \xrightarrow{L^2} u$.

Similarly

$$(4.14) \quad (\varphi, F(\eta)\eta) \rightarrow (\varphi, F(u)u).$$

Lemma 4.2.

$$(4.15) \quad \frac{1}{T} \int_{T_0}^T (\varphi, \eta |\nabla \theta|^2) dt \leq c_\delta |I|^{1/2} \|\psi\|_2 \|\psi\|_{H^1} T^{-1/2}.$$

Proof.

$$\begin{aligned} & \int_{T_0}^T (\varphi, \eta \theta'^2) dt = \int_{T_0}^T \int_I \varphi |\nabla \theta|^{3/2} \eta |\nabla \theta|^{1/2} dx dt \\ (*) & \leq \left(\int_I dx \int_{T_0}^T |\varphi|^2 |\nabla \theta|^3 dt \right)^{1/2} \left(\int_I dx \int_{T_0}^T \eta^2 |\theta'| dt \right)^{1/2} \end{aligned}$$

The second double integral is bounded as $T \rightarrow \infty$ by $(I \sup_{r \in I} \int_0^\infty |\eta^2 \theta'| dt)^{1/2} \leq |I|^{1/2} \|\psi\|_2$ by Lemma 2.6

Note that the proposition only gives a bound on the radial derivative $\hat{N} \cdot \nabla \theta$, but for radial functions it is all we need here.

The first double integral in (*) is bounded by the square root of

$$\begin{aligned} \frac{8}{\delta^3} \int_{T_0}^T dt \int_I |\varphi|^2 \eta^3 |\nabla \theta|^3 dx &\leq \frac{8}{\delta^3} \int_{T_0}^T \int \chi |\nabla \psi|^3 dx dt \\ &\leq \frac{1}{\delta^3} T^{1/4} \left(\int_{T_0}^T \|\nabla \psi\|_3^4 dt \right)^{3/4}. \end{aligned}$$

By Strichartz inequality

$$(4.16) \quad \int_{T'}^{T'+1} \|\nabla \psi\|_3^4 dt \leq c \|\psi\|_{H^1}^2,$$

so, the r.h.s. is bounded by

$$\lesssim T^{1/4} T^{3/4} = T.$$

Hence, the first double integral in (*) is bounded by $T^{1/2}$; so

$$\int_{T_0}^T (\varphi, \eta \theta'^2) dt \leq c_\delta |I|^{1/2} \|\psi\|_{H^1} \|\psi\|_{L^2} T^{1/2}.$$

□

Formally, we have:

$$(4.17) \quad \int_{\mathbb{R}^n} \eta^2 \dot{\theta} d^n x = \int |\nabla \eta|^2 + \int F(\eta^2) \eta^2 + \int \eta^2 |\nabla \theta|^2 \leq C < \infty$$

$$\frac{1}{T} \int_0^T \int_{\mathbb{R}^n} \dot{\theta} \eta^2 d^n x dt = \int_{\mathbb{R}^n} \frac{\theta(r, T)}{T} \eta^2(r, T) d^n x - \frac{1}{T} \int_{\mathbb{R}^n} \theta(r, 0) \eta^2(r, 0) d^n x$$

$$(4.18a) \quad -\frac{2}{T} \int_0^T \int_{\mathbb{R}^n} \theta \eta (2 \nabla \eta \cdot \nabla \theta + \eta \Delta \theta) d^n x dt.$$

The last term is equal to

$$(4.18b) \quad \frac{2}{T} \int_0^T \int_{\mathbb{R}^n} \eta^2 |\nabla \theta|^2 d^n x dt - \frac{2C_n}{T} \int_0^T (r^{n-1} \eta^2 \theta \theta'|_{r=0}^\infty) dt.$$

Proof. To get (4.17) we multiply equation (4.2b) by η and integrate over all space.

The last inequality follows from the energy estimate. To get (4.18a) we average over time multiplied by η , and integrate by parts in t : that, together with (12) gives (18a).

To get (4.18b) we integrate by parts again:

$$\begin{aligned} \int_0^T \int_{\mathbb{R}^n} \theta \eta (2\nabla \eta \cdot \nabla \theta + \eta \Delta \theta) d^n x dt &= \int_0^T \int_{\mathbb{R}^n} \theta (\nabla \eta^2 \cdot \nabla \theta + \eta^2 \nabla \cdot \nabla \theta) d^n x dt \\ &= \int_0^T \int_{\mathbb{R}^n} \theta \nabla (\eta^2 \nabla \theta) d^n x dt = - \int_0^T \int_{\mathbb{R}^n} \eta^2 |\nabla \theta|^2 d^n x dt + \int_0^T (C_n r^{n-1} \eta^2 \theta \theta'|_{r=0}^\infty dt. \end{aligned}$$

□

Next, we use the above heuristics, to estimate the time average of $\dot{\theta}\eta$. We repeat the above argument with $\varphi\eta$ as a weight;

Then:

$$\begin{aligned} \frac{1}{T} \int_{T_0}^T \int \varphi \dot{\theta} \eta^2 dx dt &= \int \frac{\theta(r, T)}{T} \varphi \eta_T^2 dx - \frac{1}{T} \int \theta(r, T_0) \varphi \eta_0^2 dx \\ (4.19) \quad &- \frac{2}{T} \int_{T_0}^T \int \varphi \theta \eta (2\nabla \eta \cdot \nabla \theta + \eta \Delta \theta) dx dt \end{aligned}$$

which is obtained by integration by parts in the t -variable, Recall the notation $\eta_T = \eta(r, T)$.

Now, using that

$$\eta (2\nabla \eta \cdot \nabla \theta + \eta \Delta \theta) = \nabla \cdot (\eta^2 \nabla \theta)$$

and integrating by parts in x , the last term on the r.h.s of equation (4.15) is equal to

$$\begin{aligned} \frac{2}{T} \int_{T_0}^T \int (\nabla \varphi) \cdot \theta \eta^2 \nabla \theta dx dt &+ \frac{2}{T} \int_{T_0}^T \int \varphi \eta^2 |\nabla \theta|^2 dx dt \\ (4.20) \quad &- \frac{2}{T} \int_{T_0}^T (\varphi \theta \eta^2 \nabla \theta \cdot \hat{N}|_{r=0}^\infty dt. \end{aligned}$$

So, the l.h.s. of (4.18) is equal to a sum of 5 terms.

Since φ is compactly supported the last, boundary term vanishes.

For $t = T_0$, $\theta(r, T_0)$ is bounded for r bounded, so the second term on the r.h.s. vanishes like $O(\frac{1}{T})$ as $T \rightarrow \infty$.

By Lemma (4.2) the second term of equation (4.20) vanishes like $T^{-1/2}$ as $T \rightarrow \infty$.

We are left with two terms: $\frac{\theta(r, T)}{T}$ term and φ' term.

We deal first with the $\frac{\theta(r, T)}{T}$ term:

Proposition 4.3. *For $r \in I \subset \tilde{I}$,*

$$(4.21) \quad \lim_{t \rightarrow \infty} t^{-1} \theta(r, t) \equiv -\mathcal{E}, \quad \mathcal{E} \geq 0.$$

We have

$$(4.22) \quad t^{-1} \theta(r, t) = t^{-1} \tilde{\mathcal{E}}(t) + o(t^{-3/4}) \quad t \gg 1$$

for some function $\tilde{\mathcal{E}}(t)$ independent of r .

Proof.

$$t^{-1} \theta(r, t) = t^{-1} \theta(r_0, t) + t^{-1} \int_{r_0}^r \hat{r} \cdot \nabla \theta(s, t) ds$$

$$(4.23) \quad \equiv t^{-1} \theta(r_0, t) + t^{-1} R(t)$$

and we choose: $r_0 \in I$.

Now, for $r, r_0 \in I$

$$|R(t)| \leq |r - r_0|^{1/2} \left(\int_I |\theta'(s, t)|^2 ds \right)^{1/2} \leq |I|^{1/4} \delta^{-1} O(t^{1/4})$$

by Lemma 4.2

Hence (4.23) can be written as

$$t^{-1} \theta(r, t) = t^{-1} \tilde{\mathcal{E}}(t) + o(t^{-3/4}) \quad t \gg 1.$$

with $\tilde{\mathcal{E}}(t) \equiv \theta(r_0, t)$.

This proves (4.22).

We know that for $\varphi \in H^1(I)$,

$$(4.24) \quad \int \dot{\theta} \eta \varphi = \int \varphi (-\Delta \eta) + \int \varphi F(\eta) \eta + \int \varphi \eta \theta'^2.$$

For $\varphi \in C_0^\infty(I)$, as $t \rightarrow \infty$ the r.h.s. converges to

$$\int (-\Delta \varphi) u + \int \varphi F(u) u + 0$$

in the time mean.

The first two terms converge pointwise by Lemma (4.1), and the last term converge in the mean (and therefore also for sequences) by Lemma (4.2).

Let us apply $\frac{1}{T} \int_S^T (\varphi, \cdot) dt$ to equation (4.2b):

$$\frac{1}{T} \int_S^T (\varphi, -\Delta \eta) dt + \frac{1}{T} \int_S^T (\varphi, F(\eta) \eta) dt + \frac{1}{T} \int_S^T (\varphi, \eta \theta'^2) dt = \frac{1}{T} \int_S^T (\varphi, \dot{\theta} \eta) dt.$$

We now use that if $g(t) \rightarrow g$ as $t \rightarrow \infty$, then

$$\frac{1}{T} \int_0^T g(t) dt = \frac{1}{T} \int_0^S g(t) dt + \frac{1}{T} \int_S^T g(t) dt = o\left(\frac{S}{T}\right) + g + \frac{\varepsilon|T-S|}{T}.$$

As $S \rightarrow \infty$, we can take $\varepsilon \rightarrow 0$, so that

$$\frac{1}{T} \int_0^T g(t) dt \rightarrow g.$$

So, for $\forall S \geq T_0$,

$$(4.25) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_S^T (\varphi, \dot{\theta} \eta) dt = (\varphi, F(u)u) + (-\Delta \varphi, u)$$

and using (4.15), (4.19) with $T_0 \rightarrow S$ at the lower integral limit (in t):

$$(4.26) \quad \frac{1}{T} \int_S^T \int \varphi \dot{\theta} \eta^2 dx dt = \frac{\tilde{\mathcal{E}}(T)}{T} \int \varphi \eta^2(x, T) dx + \frac{2\tilde{\mathcal{E}}(T)}{T} \int_S^T \int_I (\nabla \varphi) \cdot \nabla \theta \eta^2 dx dt + \tilde{R}(t)$$

and we want:

$$\tilde{R}(T) = O(T^{-\varepsilon}).$$

(4.27)

$$\begin{aligned} \tilde{R}(T) = & -\frac{1}{T} \int \theta(r, S) \varphi \eta_S^2 dx + \mathcal{O}(T^{-1/2}) \\ & + \int \varphi T^{-1} R(T) \eta^2(x, T) dx + 2\frac{1}{T} \int_S^T \int (\nabla \varphi) \cdot \nabla \theta R(t) \eta^2(x, t) dx dt \end{aligned}$$

(4.27) is obtained as follows: the first term comes from the second term of (4.26) (with $0 \rightarrow S$).

The second term comes from the second term of (4.19) and the estimate of Lemma (4.2).

The third term (and the forth) are obtained by replacing $\theta(r, t)$ with (4.23), and recall the definition $\theta(r_0, t) \equiv \tilde{\mathcal{E}}(t)$. Using (4.21) or (4.23), we see that the first term of $\tilde{R}(T)$ is bounded by

$$CT^{-1}T^{1/4}\|\eta\|_{L^2}^2 \rightarrow 0 \text{ as } T \rightarrow \infty, \text{ like } T^{-3/4}.$$

Similarly, the third term is vanishing like

$$T^{-3/4}.$$

Now, if we can prove that

$$\int_0^\infty \int |\nabla \varphi \cdot \nabla \theta \eta^2| dx dt < \infty$$

it will follow that the last term of (4.27) vanishes like $T^{-3/4}$.

Furthermore, by taking S large as we want, it will follow that the second term on the r.h.s. of (4.26) vanishes as $S \rightarrow \infty$, **provided** we know *a-priori* that

$$T^{-1} \tilde{\mathcal{E}}(T) \text{ is uniformly bounded in } T.$$

We extend (4.25) to $\varphi \in H^1(I)$. To this end let $f \in H^1(I)$ and $\varepsilon > 0$ be given. Then there exists $\varphi_{\varepsilon_n} \in C_0^\infty$ such that

$$\|f - \varphi_{\varepsilon_n}\|_{H^1} \leq \varepsilon/n, n = 1, 2, 3..$$

(4.28)

$$\left| \frac{1}{T} \int_S^T (\varphi_{\varepsilon_n} - \varphi_{\varepsilon_m}, \dot{\theta} \eta) dt \right| \leq |(\varphi_{\varepsilon_n} - \varphi_{\varepsilon_m}, F(u)u)| + |(\nabla(\varphi_{\varepsilon_n} - \varphi_{\varepsilon_m}), \nabla u)| + \varepsilon(T)$$

and $\varepsilon(T) \rightarrow 0$ as $T \rightarrow \infty$.

Hence

$$\left| \frac{1}{T} \int_S^T (\varphi_{\varepsilon_n} - \varphi_{\varepsilon_m}, \dot{\theta} \eta) dt \right| \leq \varepsilon(T) + \|\nabla u\|_2 \|\nabla(\varphi_{\varepsilon_n} - \varphi_{\varepsilon_m})\|_2 +$$

$$(*) \quad + \min\{\|\varphi_{\varepsilon_n} - \varphi_{\varepsilon_m}\|_6 \|F(u)u\|_{6/5}; \|F(u)u\|_2 \|\varphi_{\varepsilon_n} - \varphi_{\varepsilon_m}\|_2\} \quad (\text{in } n = 3).$$

And in dimension 1, we can replace 6 by ∞ and $6/5$ by 1. Hence the l.h.s. of (*) is bounded by

$$\varepsilon(T) + C\varepsilon/n.$$

If we now let $T \rightarrow \infty$, it follows that

$$a_n \equiv \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (\varphi_{\varepsilon_n}, \dot{\theta} \eta) dt$$

is Cauchy, and hence has a limit a .

Moreover, by (4.28)

$$(4.29) \quad a_n \rightarrow (f, F(u)u) + (\nabla f, \nabla u).$$

So, the linear operator L

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T (f, \dot{\theta}\eta) dt : f \in H^1(I) \xrightarrow{L} \mathbb{C}$$

defined on $C_0^\infty(I)$ and extended by the above limit to all $f \in H^1(I)$ satisfies the bound

$$|Lf| \leq |(f, F(u)u)| + |(\nabla f, \nabla u)| \leq c\|f\|_{H^1}\|u\|_{H^1} \leq M\|f\|_{H^1}.$$

So, it is continuous. Hence the extension of (4.25) to $\varphi \in H^1(I)$ follows.

On I , $\varphi \in C_0^\infty(I)$ implies that $\varphi u \in H^1(I)$, the extension of (4.25) to $H^1(I)$ implies that

$$\frac{1}{T} \int_S^T \varphi \dot{\theta} \eta^2 dx dt \text{ has a finite limit.}$$

So, the l.h.s. of (4.26) has a finite limit as $T \rightarrow \infty$, $\tilde{R}(T) \rightarrow 0$, and the sum of the two remaining terms on the r.h.s. of (4.26) is larger than

$$\frac{\tilde{\mathcal{E}}(T)}{T} \left[\int \varphi u^2 dx - \varepsilon_T(S) \right]$$

with $\varepsilon_T(S) \rightarrow 0$ as $T, S \rightarrow \infty$.

So, $\frac{\tilde{\mathcal{E}}(T)}{T}$ can not diverge as $T \rightarrow \infty$, and hence it has a limit $-\mathcal{E}$.

Hence, as $T \rightarrow \infty$

$$(4.30) \quad \frac{1}{T} \int_{T_0}^T \int \varphi \dot{\theta} \eta^2 dx dt \rightarrow -\mathcal{E} \int \varphi u^2 dx$$

If we choose $\varphi = \varphi_0/u$ $\varphi_0 \in C_0^\infty(I)$ we have: $\nabla \varphi = u^{-1} \nabla \varphi_0 - \varphi_0 u^{-2} \nabla u \in L^2$ since $u \geq \delta$ on I .

So, using such φ we have finally that

$$(4.31) \quad \begin{aligned} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \int \varphi_0 \dot{\theta} \eta dx dt &= -\mathcal{E} \int \varphi_0 u dx \\ &= (-\varphi_0, \Delta u) + (\varphi_0, F(u^2)u) \end{aligned}$$

or in the weak sense on test functions $\varphi_0 \in C_0^\infty(I)$:

$$(4.32) \quad -\Delta u + F(u^2)u = -\mathcal{E}u \quad r \in I.$$

Theorem 4.4. *The limiting state is a soliton for all $x \in \mathbb{R}^n, n = 1, 3$*

Proof.

A-priori, we only know that u is a soliton between two consecutive zeros of ρ . But, due to elliptic regularity, one can not have a soliton made of “patches” of excited solitons (and or zero) with the same energy.

SECTION 5 MORE GENERAL SOLUTIONS

Solutions which are completely incoming are special. In many cases, they consist of the soliton itself! A close look at the proof shows however, that the key assumption of IWC is used to prove the integrability of the (local) flux.

We can therefore obtain, by similar arguments the following more general results.

Proposition 5.1. *Let ψ_0 be as before, radial, $n = 3$. Assume the solution of the NLS has integrable incoming (and hence also outgoing) flux on any surface around the origin:*

$$\int_0^\infty dt \int_{S_R} \hat{n} \cdot (\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi})_\pm dS < \infty.$$

Here $(\cdot)_\pm$ stands for the positive / respectively negative part of

$$(\bar{\psi} \nabla \psi - \psi \nabla \bar{\psi}) \cdot \hat{n}$$

where \hat{n} is the unit normal vector to the sphere S_R of radius R around the origin. dS is the surface element on S_R .

Then

a) For any compact interval I ,

$$\| |\psi(t)| - u \|_{H^s(I)} \rightarrow 0 \text{ as } t \rightarrow \infty$$

$0 < s < 1$, and u is a soliton solution of the NLS.

b) The phase of the solution, $\theta(x, t)$ locally converges to the soliton energy, in the sense that

$$\frac{\theta(r, t)}{t} \rightarrow E, \text{ for } r\text{-fixed.}$$

Proving that the incoming/outgoing flux is integrable in time is not easy in general.

However, it should be noted that it follows from the following weaker assumption, more suitable for applications:

Proposition 5.2.

Assume ψ_0 is radial, $n = 3$.

Suppose that the solution of the NLS with this initial condition has a monotonic decreasing incoming flux, up to $L^1(dt)$ convergent part, on any sphere around the origin. Then the incoming/outgoing fluxes are absolutely integrable in time on any sphere.

The proof of proposition 5.2 follows since the total flux is integrable (not absolutely!) in time on any sphere due to L^2 boundedness and conservation.

Hence, by spherical symmetry, on any sphere, at any time the flux is either incoming or outgoing. If it is always incoming, by the integrability of the total flux, the (absolute) integrability of the incoming wave is immediate. If the flux turns from incoming to outgoing after a finite time, it stays outgoing for later times, ($+L^1(dt)$ terms) by the assumption of monotone decrease, up to $L^1(dt)$. Again, since the total flux is integrable, the purely outgoing part ($+L^1(dt)$) is absolutely integrable.

Acknowledgements

I wish to thank I. Rodnianski for very important discussions.

This work is partially supported by NSF grant DMS-0501043

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